A NOTE ON THE LINEAR SYSTEMS ON THE PROJECTIVE BUNDLES OVER ABELIAN VARIETIES

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Abstract. It is well known that for an ample line bundle L on an Abelian variety A, the linear system $|2L|$ is base point free, and $3L$ is very ample, moreover the map defined by the linear system $|2L|$ is well understood (cf. Theorem 1.1). In this paper we generalized this classical result and give a new proof using the theory developed by Pareschi and Popa in [11] (cf. Theorem 1.2).

1. Introductions

Conventions: All the varieties are assumed over $\mathbb{C}$. For a variety X and a vector bundle E on it, $\mathbb{P}_X(E)$ is defined as $Proj_{\mathcal{O}_X}(\oplus_k S^k(E^*))$, and $\mathcal{O}_{\mathbb{P}_X(E)}(1)$ denotes the anti-tautological bundle. Let $f : X \to Y$ be a morphism between two smooth projective varieties, denote by $D^b(X)$ and $D^b(Y)$ their bounded derived categories of coherent sheaves, and by Rf_* and Lf^* the derived functors of f_* and f^* respectively. For $E \in D^b(X)$, E^* denotes its dual $R\mathcal{H}om(E, \mathcal{O}_X)$, and we say that E is a sheaf if it is quasi-isomorphic to a sheaf in $D^b(X)$. If $X = X_1 \times X_2 \times ... \times X_r$ denotes the product of r varieties, then p_i denotes the projection from X to the i-th factor X_i. If A denotes an Abelian variety, then $\hat{A}$ denotes its dual $Pic^0(A)$, $\mathcal{P}$ denotes the Poincaré line bundle on $A \times \hat{A}$, and the Fourier-Mukai transform $R\Phi_{\mathcal{P}} : D^b(A) \to D^b(\hat{A})$ w.r.t. $\mathcal{P}$ is defined as

$$R\Phi_{\mathcal{P}}(\mathcal{F}) := R(p_2)_*(Lp_1^*\mathcal{F} \otimes \mathcal{P});$$

similarly $R\Psi_{\mathcal{P}} : D^b(\hat{A}) \to D^b(A)$ is defined as

$$R\Psi_{\mathcal{P}}(\mathcal{F}) := R(p_1)_*(Lp_2^*\mathcal{F} \otimes \mathcal{P})$$

For a sheaf $\mathcal{F}$ on an Abelian variety A, if $R\Phi_{\mathcal{P}}(\mathcal{F}) \cong R^0\Phi_{\mathcal{P}}(\mathcal{F})$, we say $\mathcal{F}$ satisfies IT^0. If $a : X \to A$ denotes a map to an Abelian variety, then $\mathcal{P}_a := (a \times id_{\hat{A}})^*\mathcal{P}$, and for $\mathcal{F} \in D^b(X)$, $R\Phi_{\mathcal{P}_a}(\mathcal{F})$ is defined similarly; and if $\alpha \in \hat{A}$, we often denote the line bundle $a^*\alpha \in Pic^0(X)$ by α for simplicity.

For the linear systems over an Abelian variety, the following classical results are well known.

Theorem 1.1. *Let A be an Abelian variety, and L an ample line bundle on it. Then $3L$ is very ample, $|2L|$ is base point free; and if $2L$ is not very ample, then*

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(i) *$A = A_1 \times A_2$ and $L \cong L_1 \boxtimes L_2$ where L_i is a line bundle on A_i and at least one of (A_i, L_i) is a principally polarization (*[12], [9]*);*

(ii) *moreover if A is simple and (A, L) is a principal polarization, then L is symmetric up to translation, and the map ϕ defined by $|2L|$ coincides with the quotient map $A \to A/(-1)_A$ (*[1] *Sec. 4.5, 4.8).*

In this paper, as a generalization, we prove

Theorem 1.2. *Let A be an Abelian variety, E an IT^0 vector bundle on it, $P = \mathbb{P}_A(E^*)$ with anti-tautological line bundle $\mathcal{O}_P(1)$. Then $\mathcal{O}_P(3)$ is very ample, and the linear system $|\mathcal{O}_P(2)|$ is base point free and hence defines a morphism ϕ. Moreover ϕ is not birational if and only if $A \cong A_1 \times A_2$, and $E \cong L_1 \boxtimes E_2$ where L_1 is a line bundle on A_1 and E_2 is a vector bundle on A_2 such that either $\chi(A_1, L_1) = 1$ or $\chi(A_2, E_2) = 1$.*

In particular if A is simple, then ϕ is not birational if and only if, up to translation, E is a $(-1)_A$-invariant sheaf satisfying one of the following

(i) *$\chi(A, E) = 1$, i.e., $R\Phi_{\mathcal{P}}(E) \cong \mathcal{O}_{\hat{A}}(-\hat{D})$ where $\hat{D}$ is an ample divisor on $\hat{A}$, and then $deg(\phi) = 2$ when $dim(A) \geq 2$, $deg(\phi) = 2^{rank(E)}$ when $dim(A) = 1$;*

(ii) *$E \cong \oplus^n \mathcal{O}_A(L)$ where (A, L) is a principal polarization, and then $deg(\phi) = 2$,*

meanwhile there exists an involution σ on P such that ϕ factors through the quotient map $P \to P/(\sigma)$, which fits into the following commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & P/(\sigma) \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/(-1)_A \end{array}$$

Remark 1.3. *Comparing the two properties IT^0 and ampleness, they are equivalent for a line bundle on an Abelian variety; and for a vector bundle, Debarre proved that the property IT^0 implies ampleness (*[5]*).*

Remark 1.4. *The classical proof of Theorem 1.1 is beautiful but very long. Here we not only generalized those classical results but also provided a brief proof thanks to the theory developed by Pareschi and Popa (*[11], [10]*). It is key to consider the case when A is simple, and when A is not simple, with the help of Theorem 3.6 and 3.5 in Section 3, we can also understand the morphism ϕ well (cf. Sec. 3.4).*

2. Definitions and technical results

In this section, we list some definitions and results which will be used in this paper.

Theorem 2.1 ([8], Thm. 2.2)**.** *Let A be an Abelian variety of dimension d. Then*

$$R\Psi_{\mathcal{P}} \circ R\Phi_{\mathcal{P}} = (-1)_A^*[-d], \ R\Phi_{\mathcal{P}} \circ R\Psi_{\mathcal{P}} = (-1)_{\hat{A}}^*[-d]$$

and

$$R\Phi_{\mathcal{P}} \circ (-1)_A^* \cong (-1)_{\hat{A}}^* \circ R\Phi_{\mathcal{P}}, R\Psi_{\mathcal{P}} \circ (-1)_{\hat{A}}^* \cong (-1)_A^* \circ R\Psi_{\mathcal{P}}$$

As a corollary we have

Corollary 2.2. *Let A be an Abelian variety of dimension d, E an object on $D^b(A)$, and $F = R\Phi_{\mathcal{P}}(E)^*$. Then*

$$R\Phi_{\mathcal{P}}(E) = F^*,\ \ R\Psi_{\mathcal{P}}(F) = E^*$$

Proof. We need to show the second equation. Let p_1 and p_2 be the projections from $A \times \hat{A}$ to A and $\hat{A}$ respectively. Then we have

(2.1)

$$\begin{aligned}
R\Psi_{\mathcal{P}}(F) &= R(p_1)_*(Lp_2^* R\mathcal{H}om(R\Phi_{\mathcal{P}}(E), \mathcal{O}_{\hat{A}}) \otimes \mathcal{P}) \\
&= R(p_1)_*(R\mathcal{H}om(Lp_2^* R\Phi_{\mathcal{P}}(E), \mathcal{O}_{A\times\hat{A}}) \otimes \mathcal{P}) \\
&= R(p_1)_* R\mathcal{H}om(Lp_2^* R\Phi_{\mathcal{P}}(E), \mathcal{P}) \\
&= R(p_1)_* R\mathcal{H}om(Lp_2^* R\Phi_{\mathcal{P}}(E) \otimes \mathcal{P}^{-1}, \mathcal{O}_{A\times\hat{A}}) \\
&= R\mathcal{H}om(R(p_1)_*(Lp_2^* R\Phi_{\mathcal{P}}(E) \otimes \mathcal{P}^{-1}), \mathcal{O}_A[-d]) \cdots \text{ by Grothendieck duality} \\
&= R\mathcal{H}om(R\Psi_{\mathcal{P}^{-1}} R\Phi_{\mathcal{P}}(E), \mathcal{O}_A[-d]) \\
&= R\mathcal{H}om((-1)_A^* R\Psi_{\mathcal{P}} R\Phi_{\mathcal{P}}(E), \mathcal{O}_A[-d]) \cdots \text{ by } R\Psi_{\mathcal{P}^{-1}} = (-1)_A^* R\Psi_{\mathcal{P}} \\
&= R\mathcal{H}om(E[-d], \mathcal{O}_A[-d]) = E^* \cdots \text{ by Theorem 2.1}
\end{aligned}$$

□

Definition 2.3 ([10], Def. 2.1, 2.8, 2.10, [4], Def. 2.6)**.** *Given a coherent sheaf $\mathcal{F}$ on an Abelian variety A, its i-th cohomological support locus is defined as*

$$V^i(\mathcal{F}) := \{\alpha \in Pic^0(A) | h^i(\mathcal{F} \otimes \alpha) > 0\}$$

The number $gv(\mathcal{F}) := min_{i>0}\{codim_{Pic^0(A)} V^i(\mathcal{F}) - i\}$ is called the generic vanishing index of $\mathcal{F}$, and we say $\mathcal{F}$ is a $GV-sheaf$ (resp. $M-regular\ sheaf$) if $gv(\mathcal{F}) \geq 0$ (resp. > 0).

Let X be an irregular variety equipped with a morphism to an Abelian variety $a : X \to A$. Let $\mathcal{F}$ be a sheaf on X, its i-th cohomological support locus w.r.t. a is defined as

$$V^i(\mathcal{F}, a) := \{\alpha \in Pic^0(A) | h^i(X, \mathcal{F} \otimes (a^*\alpha)) > 0\}$$

We say $\mathcal{F}$ is full w.r.t. the map a if $V^0(\mathcal{F}, a) = \hat{A}$, and is continuously globally generated (CGG) w.r.t. a if the sum of the evaluation maps

$$ev_U : \oplus_{\alpha\in U} H^0(\mathcal{F} \otimes \alpha) \otimes (\alpha^{-1}) \to \mathcal{F}$$

is surjective for any open set $U \subset \hat{A}$.

Let's recall the following results due to Pareschi and Popa.

Proposition 2.4 ([10], Prop. 3.1)**.** *Let $\mathcal{F}$ be a GV-sheaf and H a locally free IT^0 sheaf on an Abelian variety A. Then $\mathcal{F} \otimes H$ satisfies IT^0.*

Proposition 2.5 ([10], Cor. 5.3)**.** *An M-regular sheaf on an Abelian variety is CGG.*

Proposition 2.6 ([11], Prop. 2.12)**.** *Let X be an irregular variety equipped with a morphism to an Abelian variety $a : X \to A$. Let F be a coherent sheaf and L a line bundle on X. Suppose that both F and L are continuously globally generated w.r.t. a. Then $F \otimes L \otimes \alpha$ is globally generated for all $\alpha \in Pic^0(A)$.*

Theorem 2.7 ([11], Thm. 5.1)**.** *Let F be a GV-sheaf on an Abelian variety A. Then the following conditions are equivalent:*

(a) *F is M-regular.*
(b) *For every locally free IT^0 sheaf H on A, and for every Zariski open set $U \subset \hat{A}$, the sum of the multiplication of maps of global sections*

$$\oplus_{\alpha \in U} H^0(A, F \otimes \alpha) \otimes H^0(A, H \otimes \alpha^{-1}) \to H^0(A, F \otimes H)$$

is surjective.

Remark 2.8. *Modifying the proof of Thm. 5.1 in* [11], *we can show that for a GV-sheaf F and a locally free IT^0 sheaf H on A, the sum of the multiplication of maps of global sections*

$$\oplus_{\alpha \in \hat{A}} H^0(A, F \otimes \alpha) \otimes H^0(A, H \otimes \alpha^{-1}) \to H^0(A, F \otimes H)$$

is surjective.

As a corollary of the theorem above, we get an interesting result.

Lemma 2.9. *Let E be an IT^0 vector bundle on an Abelian variety A. Suppose that $\chi(A, E) = 1$ and that E is a $(-1)_A$-invariant sheaf. Let $\varphi_E : E \to (-1)_A^* E$ be the corresponding isomorphism. Then naturally $\varphi_E \otimes \varphi_E : E \otimes E \to (-1)_A^*(E \otimes E)$ shows that $E \otimes E$ is a $(-1)_A$-invariant sheaf, thus $(-1)_A$ induces an action $(-1)_A^*$ on $H^0(A, E \otimes E)$, moreover if denoting its invariant (resp. anti-invariant) subspace by $H^0(A, E \otimes E)^+$ (resp. $H^0(A, E \otimes E)^-$), then we have*

$$H^0(A, S^2E) \cong H^0(A, E \otimes E)^+ \text{ and } H^0(A, \wedge^2 E) \cong H^0(A, E \otimes E)^-$$

Proof. For every $\alpha \in \hat{A}$, we can assume $H^0(A, E \otimes \alpha) = span\{e_\alpha\}$ since $h^0(A, E \otimes \alpha) = 1$. Applying Theorem 2.8, $H^0(A, E \otimes E)$ is spanned by $\{e_\alpha \otimes e_{\alpha^{-1}}\}_{\alpha \in \hat{A}}$, and hence $H^0(A, S^2E)$ (resp. $H^0(A, \wedge^2 E)$)is spanned by $\{e_\alpha \otimes e_{\alpha^{-1}} + e_\alpha \otimes e_{\alpha^{-1}}\}_{\alpha \in \hat{A}}$ (resp. $\{e_\alpha \otimes e_{\alpha^{-1}} - e_\alpha \otimes e_{\alpha^{-1}}\}_{\alpha \in \hat{A}}$). Since $(-1)_A^* e_\alpha$ is a section of $(-1)_A^*(E \otimes \alpha) \cong E \otimes \alpha^{-1}$ and $H^0(A, E \otimes \alpha^{-1})$ is spanned by $e_{\alpha^{-1}}$, we can write $(-1)_A^* e_\alpha = c e_{\alpha^{-1}}$ with $c \neq 0$, and then $(-1)_A^* e_{\alpha^{-1}} = \frac{1}{c} e_\alpha$, hence $(-1)_A^*(e_\alpha \otimes e_{\alpha^{-1}}) = e_{\alpha^{-1}} \otimes e_\alpha$. Finally we find that

$$(-1)_A^*(e_\alpha \otimes e_{\alpha^{-1}} + e_\alpha \otimes e_{\alpha^{-1}}) = e_\alpha \otimes e_{\alpha^{-1}} + e_\alpha \otimes e_{\alpha^{-1}}$$

and

$$(-1)_A^*(e_\alpha \otimes e_{\alpha^{-1}} - e_\alpha \otimes e_{\alpha^{-1}}) = -(e_\alpha \otimes e_{\alpha^{-1}} - e_\alpha \otimes e_{\alpha^{-1}})$$

So we are done. □

Here we make a simple but very useful remark, which is probably well known to experts, but we are not able to find a reference.

Theorem 2.10. *Let X and Y be two normal projective varieties, and $\mathcal{L}$ a line bundle on $X \times Y$. Assume $E = (p_2)_* \mathcal{L}$ is a vector bundle and put $P = \mathbb{P}_Y(E)$. Then there exists an open set $U \subset Y$ such that $\mathbb{P}_U(E)$ parametrizes the divisors in $|\mathcal{L}_y|, y \in U$, correspondingly we get the universal family $\mathcal{D}_U \subset X \times U \to U$. Denote the closure of $\mathcal{D}_U$ in $X \times P$ by $\mathcal{D}$, which is embedded in $X \times P$ as a divisor. Then we have*

$$\mathcal{D} \equiv p^* \mathcal{L} \otimes q^* \mathcal{O}_P(1)$$

where p, q denote the two projections $p : X \times P \to X \times Y$, $q : X \times P \to P$.

Proof. Let $\{U_\alpha\}_{\alpha\in I}$ be an affine cover of Y. Assume E is of rank n and $E(U_\alpha) = \mathcal{O}_Y(U_\alpha)(s_\alpha^1, s_\alpha^2, ..., s_\alpha^n)$ where $s_\alpha^i \in H^0(X \times U_\alpha, \mathcal{L}) \cong E(U_\alpha)$, and the divisors (s_α^i) have no common vertical components (where a vertical divisor means an effective divisor pulled back via p_2); and denote by $x_\alpha^i \in E^*(U_\alpha), i = 1, 2, ..., n$ the dual basis of $\{s_\alpha^i\}_{i=1,2,...,n}$. Then $x_\alpha^1, ...x_\alpha^n$ can be seen as a coordinate of $E(U_\alpha)$.

The equation $\sum_i x_\alpha^i s_\alpha^i = 0$ defines a divisor $\mathcal{D}'_\alpha \subset X \times \mathbb{P}_{U_\alpha}(E_{U_\alpha})$, which coincides with $\mathcal{D}$ if restricted to the open set $X \times \mathbb{P}_{U_\alpha \cap U}(E)$. We patch together the $\mathcal{D}'_\alpha$'s and get a divisor $\mathcal{D}'$. By assumption, the divisor $\mathcal{D}'$ has no vertical part, hence $\mathcal{D}' = \mathcal{D}$. By construction, $\mathcal{D}' \equiv p^*\mathcal{L} \otimes q^*\mathcal{O}_P(1)$, and we are done. □

3. Proof of Theorem 1.2

This section is devoted to prove Theorem 1.2.

Let $\hat{P} = \mathbb{P}_{\hat{A}}(R\Phi_{\mathcal{P}}(E))$, $n = rank(E)$ and $m = rank(R\Phi_{\mathcal{P}}(E))$. $\pi : P \to A$ and $\hat{\pi} : \hat{P} \to \hat{A}$ denote the natural projections, which coincide with their Albanese maps respectively. We denote by $\tilde{\mathcal{P}}$ the pull-back $(\pi \times \hat{\pi})^*\mathcal{P}$ of the Poincaré bundle on $A \times \hat{A}$.

By Corollary 2.2, we conclude that

$$(p_2)_*(p_1^*\mathcal{O}_P(1) \otimes \mathcal{P}_\pi) \cong R\Phi_{\mathcal{P}}(\pi_*\mathcal{O}_P(1)) \cong R\Phi_{\mathcal{P}}(E) \cong R^0\Phi_{\mathcal{P}}(E)$$

and

$$(p_1)_*(p_2^*\mathcal{O}_{\hat{P}}(1) \otimes \mathcal{P}_{\hat{\pi}}) \cong R\Psi_{\mathcal{P}}(\hat{\pi}_*\mathcal{O}_{\hat{P}}(1)) \cong R\Psi_{\mathcal{P}}(R\Phi_{\mathcal{P}}(E)^*) \cong E^*$$

where the maps p_1, p_2, in the first equation denote the projections from $P \times \hat{A}$ to P and $\hat{A}$ respectively, and in the second equation denote the projections from $A \times \hat{P}$ to A and $\hat{P}$ respectively. We identify P (resp. $\hat{P}$) with the Hilbert scheme parametrizing the divisors in $\{|\mathcal{O}_{\hat{P}}(1) \otimes \hat{\alpha}||\hat{\alpha} \in Pic^0(\hat{P}) = A\}$ (resp. $\{|\mathcal{O}_P(1) \otimes \alpha||\alpha \in Pic^0(P) = \hat{A}\}$), and denote by $\mathcal{Y} \subset P \times \hat{P}$ the universal family. Using Theorem 2.10 we get

$$\mathcal{Y} \equiv p_1^*\mathcal{O}_P(1) \otimes p_2^*\mathcal{O}_{\hat{P}}(1) \otimes \tilde{\mathcal{P}} \tag{3.1}$$

Immediately from Eq. 3.1, it follows that for $x \in P$

$$\mathcal{Y}_x \equiv (p_1^*\mathcal{O}_P(1) \otimes p_2^*\mathcal{O}_{\hat{P}}(1) \otimes \tilde{\mathcal{P}})_x \equiv \mathcal{O}_{\hat{P}}(1) \otimes \mathcal{P}_{\pi(x)}$$

Since $R^i\hat{\pi}_*\mathcal{O}_{\hat{P}}(1) = 0$ for $i > 0$ and $\hat{\pi}_*\mathcal{O}_{\hat{P}}(1)$ is an IT^0 sheaf, we have $h^i(\hat{P}, \mathcal{O}_{\hat{P}}(1) \otimes \mathcal{P}_a) = 0$ for $i > 0, a \in A$, and remark that

- ♢ identifying a divisor in $|\mathcal{O}_P(1) \otimes \alpha|, \alpha \in \hat{A}$ with a point in $\hat{P}$, for every $x \in P$, $\mathcal{Y}_x \equiv \mathcal{O}_{\hat{P}}(1) \otimes \mathcal{P}_{\pi(x)}$ parametrizes all those divisors passing through x; and for every divisor $Y \equiv \mathcal{O}_{\hat{P}}(1) \otimes \mathcal{P}_a, a \in A$ (equivalently, topologically equivalent to $\mathcal{Y}_x$ for some $x \in P$), there exists unique $y \in P$ such that $\mathcal{Y}_y = Y$.

We conclude that for $x, y \in P$,

$$\mathcal{Y}_x \equiv \mathcal{Y}_y \Leftrightarrow \pi(x) = \pi(y), \mathcal{Y}_x = \mathcal{Y}_y \Leftrightarrow x = y$$

And we can write that

$$\mathcal{Y}_x = \mathcal{H}_x + \mathcal{V}_x \text{ and } \mathcal{V}_x = \mathcal{V}_x^1 + ... + \mathcal{V}_x^r \tag{3.2}$$

where $\mathcal{H}_x$ is the horizontal part (if $m = 1$ then $\mathcal{H}_x = \emptyset$), $\mathcal{V}_x = \hat{\pi}^*V_x$ is the vertical part ($\mathcal{V}_x = \emptyset$ if $\mathcal{Y}_x$ is irreducible), and the $\mathcal{V}_x^i = \hat{\pi}^*V_x^i$'s are the reduced and irreducible vertical components (two of them may equal). Here we remark that

♡ as the divisor V_x varies continuously in $\hat{A}$, the divisor $\mathcal{Y}_x = \mathcal{H}_x + \mathcal{V}_x$ varies in the same topological class, correspondingly x varies continuously in P by ♢, so if x is general, we can assume that the divisors $V_x^1, ..., V_x^r, (-1)^*_{\hat{A}} V_x^1, ..., (-1)^*_{\hat{A}} V_x^r$ are distinct to each other.

Remark 3.1. *Suppose that for general $x \in P$, $\mathcal{Y}_x$ is reducible. Then there exist an open set $U \subset P$ and two divisors $\mathcal{H}$ and $\mathcal{V}$ on $U \times \hat{P}$, such that for $x \in U$, $\mathcal{H}_x$ (resp. $\mathcal{V}_x$) defined above coincides with the fiber of $\mathcal{H}$ (resp. $\mathcal{V}$) over x. We also denote the closure of the two divisors in $P \times \hat{P}$ by $\mathcal{H}$ and $\mathcal{V}$. Then since $\mathcal{Y}_x$ is a divisor in $\hat{P}$ for every $x \in P$, so $\mathcal{Y}$ contains no component which is the pull-back of a divisor in P via the projection $P \times \hat{P} \to P$, thus $\mathcal{Y} = \mathcal{H} + \mathcal{V}$.*

3.1. **Base points and the degree of ϕ.**

Proposition 3.2. *$|\mathcal{O}_P(2)|$ is base point free, and $\mathcal{O}_P(3)$ is very ample.*

Proof. Fixing a point $x \in P$. Since $\mathcal{Y}_x$ is a divisor on P, for a general $\alpha \in \hat{A}$, we can find $H_\alpha \in |\mathcal{O}_P(1) \otimes \alpha|$ and $H_{\alpha^{-1}} \in |\mathcal{O}_P(1) \otimes \alpha^{-1}|$ such that neither H_α nor $H_{\alpha^{-1}}$ is contained in $\mathcal{Y}_x$, which means x is not contained in $H_\alpha + H_{\alpha^{-1}}$ by ♢. Then we can see that $|\mathcal{O}_P(2)|$ has no base point.

Let $x \in P$, denote by I_x its ideal sheaf. Consider the following exact sequence

$$0 \to I_x \otimes \mathcal{O}_P(1) \to \mathcal{O}_P(1) \to \mathbb{C}(x) \to 0$$

Applying π_* to the sequence above we obtain an exact sequence on A

$$0 \to \pi_*(I_x \otimes \mathcal{O}_P(1)) \to E \to \mathbb{C}(\pi(x)) \to 0$$

Then applying $R\Phi_{\mathcal{P}}$ to the sequence above, we get

$$0 \to R^0\Phi_{\mathcal{P}}(\pi_*(I_x \otimes \mathcal{O}_P(1))) \to R^0\Phi_{\mathcal{P}}(E) \to \mathcal{P}_{\pi(x)} \to R^1\Phi_{\mathcal{P}}(\pi_*(I_x \otimes \mathcal{O}_P(1))) \to 0$$

and that $R^i\Phi_{\mathcal{P}}(\pi_*(I_x \otimes \mathcal{O}_P(1))) = 0$ if $i > 1$ since then $R^i\Phi_{\mathcal{P}}(E) = R^i\Phi_{\mathcal{P}}(\mathbb{C}(\pi(x))) = 0$.

Restricting $\mathcal{Y}_x$ to a general fiber of $\hat{P} \to A$, it is a hyperplane, so

$$rank(R^0\Phi_{\mathcal{P}}(\pi_*(I_x \otimes \mathcal{O}_P(1)))) = rank(R^0\Phi_{\mathcal{P}}(E)) - 1$$

And since $rank(\mathcal{P}_{\pi(x)}) = 1$, we conclude that $codim_{\hat{A}}(Supp(R^1\Phi_{\mathcal{P}}(\pi_*(I_x \otimes \mathcal{O}_P(1))))) \geq 1$, so $\pi_*(I_x \otimes \mathcal{O}_P(1))$ is a GV-sheaf. Proposition 2.4 tells that $\pi_*(I_x \otimes \mathcal{O}_P(1)) \otimes E$ satisfies IT^0, thus is CGG by Proposition 2.5. Since both the natural maps

$$\pi_*(I_x \otimes \mathcal{O}_P(1)) \otimes E \to \pi_*(I_x \otimes \mathcal{O}_P(2))$$

and

$$\pi^*(\pi_*(I_x \otimes \mathcal{O}_P(2))) \to I_x \otimes \mathcal{O}_P(2)$$

are surjective, $I_x \otimes \mathcal{O}_P(2)$ is CGG w.r.t. π. Note that $\mathcal{O}_P(1)$ is CGG w.r.t. π since $\pi_*\mathcal{O}_P(1) \cong E$ is CGG and $\pi^*E \to \mathcal{O}_P(1)$ is surjective. Using Proposition 2.6, it follows that $I_x \otimes \mathcal{O}_P(3)$ is globally generated. Therefore, the line bundle $\mathcal{O}_P(3)$ is very ample. □

Lemma 3.3. *Let $x, y \in P$ be two distinct points. Write that $\mathcal{Y}_x = \mathcal{H}_x + \mathcal{V}_x$ and $\mathcal{Y}_y = \mathcal{H}_y + \mathcal{V}_y$ as in 3.2. Then the following conditions are equivalent*

(a) *$|\mathcal{O}_P(2)|$ fails to separate x, y;*
(b) *$\mathcal{H}_x = \mathcal{H}_y$ and $Supp(V_x + (-1)^*_{\hat{A}} V_x) = Supp(V_y + (-1)^*_{\hat{A}} V_y)$.*

Proof. First we show $(a) \Rightarrow (b)$. Now suppose that $|\mathcal{O}_P(2)|$ fails to separate x, y.

We claim that $\mathcal{H}_x = \mathcal{H}_y$ if $m > 1$. Indeed, otherwise for a general $\alpha \in \hat{A}$, we can find $H_\alpha \in |\mathcal{O}_P(1) \otimes \alpha|, H_{\alpha^{-1}} \in |\mathcal{O}_P(1) \otimes \alpha^{-1}|$ such that $H_\alpha \in \mathcal{Y}_x$ and neither H_α nor $H_{\alpha^{-1}}$ is contained in $\mathcal{Y}_y$. So $H_\alpha + H_{\alpha^{-1}} \in |\mathcal{O}_P(2)|$ contains x but y by $\diamondsuit$, therefore $|\mathcal{O}_P(2)|$ separates x, y.

Let $\alpha \in V_x$. Then $x \in Bs|\mathcal{O}_P(1) \otimes \alpha|$. Let $\mathcal{V}_x^i = \hat{\pi}^* V_x^i$ be an irreducible component of $\mathcal{Y}_x$. Then for any $\alpha \in V_x^i$, any $H_\alpha \in |\mathcal{O}_P(1) \otimes \alpha| \subset \mathcal{V}_x^i$ and $H_{\alpha^{-1}} \in |\mathcal{O}_P(1) \otimes \alpha^{-1}| \subset \hat{\pi}^*(-1)_{\hat{A}}^* V_x^i$, since $x \in H_\alpha + H_{\alpha^{-1}}$, we have $y \in H_\alpha + H_{\alpha^{-1}}$, so either $H_\alpha \in \mathcal{Y}_y$ or $H_{\alpha^{-1}} \in \mathcal{Y}_y$. Then we conclude that $\mathcal{V}_x^i \subset \mathcal{Y}_y$ or $\hat{\pi}^*(-1)_{\hat{A}}^* V_x^i \subset \mathcal{Y}_y$ since $\mathcal{V}_x^i$ is irreducible, so it follows that

$$Supp(V_x) \subset Supp(V_y + (-1)_{\hat{A}}^* V_y) \textit{ and } Supp((-1)_{\hat{A}}^* V_x) \subset Supp(V_y + (-1)_{\hat{A}}^* V_y)$$

thus

$$Supp(V_x + (-1)_{\hat{A}}^* V_x) \subset Supp(V_y + (-1)_{\hat{A}}^* V_y)$$

In the same way, we show that $Supp(V_x + (-1)_{\hat{A}}^* V_x) \supset Supp(V_y + (-1)_{\hat{A}}^* V_y)$, so one direction follows.

Now we show $(b) \Rightarrow (a)$, so assume (b).

Note that $\pi_*(I_x \otimes \mathcal{O}_P(1))$ is a GV-sheaf which is proved during the proof of Proposition 3.2. Again since the natural map

$$\pi_*(I_x \otimes \mathcal{O}_P(1)) \otimes E \to \pi_*(I_x \otimes \mathcal{O}_P(2))$$

is surjective, using Remark 2.8, we conclude that $H^0(A, I_x \otimes \mathcal{O}_P(2))$ is spanned by the elements in the set

$$\{e_\alpha \otimes f_{\alpha^{-1}} \in H^0(P, \mathcal{O}_P(2)) | \alpha \in \hat{A}, e_\alpha \in H^0(P, I_x \otimes \mathcal{O}_P(1) \otimes \alpha), f_{\alpha^{-1}} \in H^0(P, \mathcal{O}_P(1) \otimes \alpha^{-1})\}$$

On the other hand reversing the argument when proving $(a) \Rightarrow (b)$, we can prove that

$$x \in H_\alpha + H_{\alpha^{-1}} \Rightarrow y \in H_\alpha + H_{\alpha^{-1}}$$

where $H_\alpha \in |\mathcal{O}_P(1) \otimes \alpha|, H_{\alpha^{-1}} \in |\mathcal{O}_P(1) \otimes \alpha^{-1}|$. Then by the fact that

$$(e_\alpha \otimes f_{\alpha^{-1}}) \in |\mathcal{O}_P(1) \otimes \alpha| + |\mathcal{O}_P(1) \otimes \alpha^{-1}| \subset |\mathcal{O}_P(2)|$$

we conclude that every divisor in $|\mathcal{O}_P(2)|$ containing x contains y. So this direction follows, and we are done. $\square$

Corollary 3.4. *Let x be a general point, and write that $\mathcal{Y}_x = \mathcal{H}_x + \mathcal{V}_x^1 + ... + \mathcal{V}_x^r$ as in 3.2. Then the degree of ϕ is 2^r. In particular, ϕ is birational if and only if for a general $x \in P$, $\mathcal{Y}_x$ is irreducible.*

Proof. By $\heartsuit$, we can assume the divisors $V_x^1, ..., V_x^r, (-1)_{\hat{A}}^* V_x^1, ..., (-1)_{\hat{A}}^* V_x^r$ are distinct to each other. For a point $y \in P$ distinct to x, since $V_x + (-1)_{\hat{A}}^* V_x$ is reduced, using Lemma 3.3, we know that $|\mathcal{O}_P(2)|$ fails to separate x and y if and only if

$$V_x + (-1)_{\hat{A}}^* V_x = V_y + (-1)_{\hat{A}}^* V_y$$

equivalently

$$V_y = ((-1)_{\hat{A}}^{\epsilon_1})^* V_x^1 + ... + ((-1)_{\hat{A}}^{\epsilon_r})^* V_x^r, \ \epsilon_i \in \{0, 1\}, i = 1, 2, ..., r$$

On the other hand, for every choice $\epsilon_i \in \{0, 1\}, i = 1, 2, ..., r$, since $((-1)_{\hat{A}}^{\epsilon_1})^* V_x^i - V_x^i \in Pic^0(\hat{A})$, there exists $\alpha \in Pic^0(\hat{P})$ such that

$$\mathcal{H}_x + \hat{\pi}^*((-1)_{\hat{A}}^{\epsilon_1})^* V_x^1 + ... + \hat{\pi}^*((-1)_{\hat{A}}^{\epsilon_r})^* V_x^r \equiv \mathcal{Y}_x + \alpha$$

thus there exists unique $y \in P$ such that

$$\mathcal{Y}_y = \mathcal{H}_x + \hat{\pi}^*((-1)_{\hat{A}}^{\epsilon_1})^* V_x^1 + ... + \hat{\pi}^*((-1)_{\hat{A}}^{\epsilon_r})^* V_x^r$$

Then we conclude that $deg(\phi) = 2^r$, and the remaining assertion follows easily. $\square$

3.2. **The reducibility of the general hyperplane section.**

Theorem 3.5. *For a general $x \in P$, if $\mathcal{Y}_x$ has non-trivial vertical part, then $A \cong A_1 \times A_2$ (A_1, A_2 maybe a point), and there exists a line bundle L_1 on A_1 and a vector bundle E_2 on A_2 such that $E \cong L_1 \boxtimes E_2$ and that $\chi(A_1, L_1) = 1$ or $\chi(A_2, E_2) = 1$.*

The converse is also true.

Proof. If $m = \chi(E) = 1$, then $R^0\Phi_{\mathcal{P}}(E)$ is a line bundle on $\hat{A}$. Let $A_1 = pt$, $A_2 = A$, L_1 the trivial line bundle on A_1 and $E_2 = E$ on A_2. Then we are done.

Now assume $m > 1$. By assumption for a general x, we can write $\mathcal{Y}_x = \mathcal{H}_x + \mathcal{V}_x$ as in 3.2, correspondingly write that $\mathcal{Y} = \mathcal{H} + \mathcal{V}$ as in Remark 3.1. In the following, for every $x \in P$, the divisors $\mathcal{H}_x$ and $\mathcal{V}_x$ denote the fibers of $\mathcal{H}$ and $\mathcal{V}$ over x respectively.

Let $a \in A$ be general. First we claim that for two general points $x_1, x_2 \in P$ over a, $\mathcal{H}_{x_1} \equiv \mathcal{H}_{x_2}$. Indeed, for a general point x_0 fixed over a, we can define a morphism $\pi^{-1}a \to A$ via $x \mapsto \mathcal{H}_x - \mathcal{H}_{x_0}$ by identifying $\mathcal{H}_x - \mathcal{H}_{x_0}$ with an element in $Pic^0(\hat{A}) \cong A$. This map must be constant since $\pi^{-1}a$ is rational. So this assertion is true. In turn we conclude that

$$|\mathcal{O}_{\hat{P}}(1) \otimes a| = |\mathcal{H}_x + \mathcal{V}_x| = |\mathcal{H}_x| + |\mathcal{V}_x| \text{ where } x \in P \text{ is a general point over } a \tag{3.3}$$

and that $|\mathcal{H}_x| = \mathcal{H}_x$ or $|\mathcal{V}_x| = \mathcal{V}_x$ holds true due to Bertini's theorem. By semicontinuity, we conclude that both $h^0(\hat{P}, \mathcal{H}_x)$ and $h^0(\hat{P}, \mathcal{V}_x)$ are invariant as x varies in P, and one of them is equal to m and the other is equal to 1.

Therefore, for a general $a_0 \in A$ fixed, we can define two rational maps $\iota_1, \iota_2 : A \dashrightarrow A$ by $\iota_1 : a \mapsto \mathcal{H}_x - \mathcal{H}_{x_0}$ and $\iota_2 : a \mapsto \mathcal{V}_x - \mathcal{V}_{x_0}$ where x, x_0 are two general points over a, a_0 respectively, which are well defined and extend to two morphisms. Denote their images by A_1 and A_2, which are two sub-torus passing through $0 \in A$ hence subgroups of A.

Let $a \in A_1 \cap A_2$, then $a^{-1} \in A_1 \cap A_2$. We can find $x_1, x_2 \in P$ such that $\iota_1(\pi(x_1)) = a, \iota_2(\pi(x_2)) = a^{-1}$. It follows that $|\mathcal{H}_{x_1}| = |\mathcal{H}_{x_0} \otimes a| \neq \emptyset$ and $|\mathcal{V}_{x_2}| = |\mathcal{V}_{x_0} \otimes a^{-1}| \neq \emptyset$, and therefore,

$$|\mathcal{H}_{x_0} \otimes a| + |\mathcal{V}_{x_0} \otimes (a^{-1})| = |\mathcal{H}_{x_1}| + |\mathcal{V}_{x_2}| \subset |\mathcal{O}_{\hat{P}}(1) \otimes a_0|$$

Note that a general divisor in $|\mathcal{H}_{x_0}|$ is irreducible since a_0 is general, $h^0(\hat{P}, \mathcal{H}_{x_0}) = h^0(\hat{P}, \mathcal{H}_{x_1})$, and $\mathcal{H}_{x_0} \sim_{num} \mathcal{H}_{x_1}$. So Eq. 3.3 implies that $\mathcal{H}_{x_0} \equiv \mathcal{H}_{x_1}$, thus $a = 0$. Therefore, $A_1 \cap A_2 = \{0\}$.

On the other hand, for general $a \in A$ and $x \in P$ over a, by definition it follows that

$$|\mathcal{H}_x| = |\mathcal{H}_{x_0} \otimes \iota_1(a)| \neq \emptyset, |\mathcal{V}_x| = |\mathcal{V}_{x_0} \otimes \iota_2(a)| \neq \emptyset$$

hence $|\mathcal{O}_{\hat{P}}(1) \otimes a| = |\mathcal{H}_{x_0} \otimes \iota_1(a)| + |\mathcal{V}_{x_0} \otimes \iota_2(a)|$. By $|\mathcal{O}_{\hat{P}}(1) \otimes a_0| = |\mathcal{H}_{x_0}| + |\mathcal{V}_{x_0}|$, we conclude that $a - a_0 = \iota_1(a) + \iota_2(a)$, consequently $A \cong A_1 \times A_2$.

Let $\hat{A}_i = Pic^0(A_i)$ and denote by $\mathcal{P}_i$ the pull-back of the Poincaré bundle on $A_i \times \hat{A}_i$ via the map $A_i \times \hat{A} \to A_i \times \hat{A}_i$. By the analysis above we conclude that

$$R\Psi_{\mathcal{P}}(\hat{\pi}_* \mathcal{O}_{\hat{P}}(1) \otimes a_0) \cong R^0\Psi_{\mathcal{P}_1}(\hat{\pi}_* \mathcal{O}_{\hat{P}}(\mathcal{H}_{x_0})) \boxtimes R^0\Psi_{\mathcal{P}_2}(\hat{\pi}_* \mathcal{O}_{\hat{P}}(\mathcal{V}_{x_0}))$$

Then recalling that $|\mathcal{H}_x| = \mathcal{H}_x$ or $|\mathcal{V}_x| = \mathcal{V}_x$ holds, we conclude that at least one of $R^0\Psi_{\mathcal{P}_1}(\hat{\pi}_* \mathcal{O}_{\hat{P}}(\mathcal{H}_{x_0}))$ and $R^0\Psi_{\mathcal{P}_2}(\hat{\pi}_* \mathcal{O}_{\hat{P}}(\mathcal{V}_{x_0}))$ is a line bundle. Then since $R\Psi_{\mathcal{P}}(\hat{\pi}_* \mathcal{O}_{\hat{P}}(1) \otimes a_0) \cong t^*_{-a_0} E^*$ where t_{-a_0} is the translation by $-a_0$, so one direction is completed.

The converse assertion is obvious, so this theorem is true. □

3.3. **The map ϕ.** We distinguish the two cases $m = 1$ and $m > 1$.

Case $m = 1$: In this case, for every $x \in P$, $\mathcal{Y}_x$ is a divisor V_x on $\hat{A}$, hence ϕ is surely not birational by Corollary 3.4, and $E = R\Psi_{\mathcal{P}}(\mathcal{O}_{\hat{A}}(V))^*$ for some ample divisor V on $\hat{A}$. Find a suitable $a_0 \in A$ such that $\mathcal{O}_{\hat{A}}(V) \otimes a_0$ is symmetric. Then its Fourier-Mukai transform $R\Psi_{\mathcal{P}}(\mathcal{O}_{\hat{A}}(V) \otimes a_0) \cong t^*_{-a_0} E^*$ is a $(-1)_A$-invariant sheaf. Replacing E by $t^*_{-a_0} E$, we can assume E is a $(-1)_A$-invariant sheaf, so $(-1)_A$ induces an action $(-1)^*_A$ on $H^0(A, E \otimes E)$. Note that $H^0(P, \mathcal{O}_P(2)) \cong H^0(A, S^2E)$. By Lemma 2.9, $(-1)^*_A$ is identity on $H^0(A, S^2E)$, hence ϕ factors through an involution σ making the following commutative diagram hold

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi"'] & P/(\sigma) \arrow[d, "\pi'"'] \\ A \arrow[r] & A/((-1)_A) \end{tikzcd}$$

In particular, if A is simple, then Corollary 3.4 gives that $deg(\phi) = 2$ if $dim(A) > 1$ since then a general V_x is irreducible, and $deg(\phi) = 2^n$ if $dim(A) = 1$ since then $deg(V_x) = rank(E) = n$.

Case $m > 1$: In this case, using Corollary 3.4, immediately it follows that ϕ is not birational if and only if for a general $x \in P$, $\mathcal{Y}_x$ is reducible.

If A is simple and ϕ is not birational, applying Theorem 3.5, we conclude that $A \cong A_1 \times A_2$ and hence either A_1 or A_2 is a point. So we can assume $A_1 = A$, L_1 be a line bundle on A_1 such that (A, L_1) is a principally polarization, $A_2 = pt$ and E_2 is a vector bundle of rank > 1. Therefore $P \cong A \times \mathbb{P}^{n-1}$, then reducing to the case when $m = 1$, we can show ϕ is of degree 2.

In conclusion, we finally proved Theorem 1.2.

3.4. **The degree of ϕ when A is not simple.** When A is not simple, to study the degree of ϕ, it suffices to consider the case when $m = 1$, i.e., $R\Phi_{\mathcal{P}}(E)$ is quasi-isomorphic to a line bundle. This is completely clear thanks to the following classical result.

Theorem 3.6 ([1] Sec. 3.4)**.** *Let A be an Abelian variety, D an ample divisor on A. Then the linear system $|D|$ can be written as $|D| = |D_1| + D_2 + ... + D_r$ where*

(i) *$|D_1|$ is the moving part;*
(ii) *$A \cong A_1 \times A_2 \times ... \times A_r$ where $A_i, i = 2, 3, ..., r$ is simple;*
(iii) *for $i = 1, 2, ..., r$, $D_i \equiv p_i^* B_i$ where B_i is a divisor on A_i, and for $i = 2, ..., r$, (A_i, B_i) is a principal polarization.*

Moreover if $dim(A_1) > 1$, then a general element in $|B_1|$ is irreducible.

Corollary 3.7. *Assume that $R\Phi_{\mathcal{P}}(E) = \mathcal{O}_{\hat{A}}(-\hat{D})$ for some ample divisor $\hat{D}$ on $\hat{A}$, and the linear system $|\hat{D}| = |\hat{D}_1| + \hat{D}_2 + ... + \hat{D}_r$ as in Theorem 3.6, where $\hat{A} \cong \hat{A}_1 \times \hat{A}_2 \times ... \times \hat{A}_r$ and $\hat{D}_i = p_i^*\hat{B}_i$. Then we have*

(1) *if $\hat{A}_1$ is an elliptic curve, then $deg(\phi) = 2^{deg(\hat{B}_1)+r-1}$;*
(2) *if $dim(\hat{A}_1) > 1$, then $deg(\phi) = 2^r$*

Proof. For $a \in A$, there exists $\alpha \in \hat{A}$ such that $\mathcal{O}_{\hat{A}}(\hat{D}) \otimes \mathcal{P}_a \equiv t_\alpha^*\hat{D}$, hence

$$|\mathcal{O}_{\hat{A}}(\hat{D}) \otimes \mathcal{P}_a| = |t_\alpha^*\hat{D}_1| + t_\alpha^*\hat{D}_2 + ... + t_\alpha^*\hat{D}_r$$

For $x \in P$, the divisor $\mathcal{V}_x \in |\mathcal{O}_{\hat{A}}(\hat{D}) \otimes \mathcal{P}_{\pi(x)}|$. Then our assertions follow from Corollary 3.4. □

3.5. **Applications.** The results and the ideas of the proof probably find their applications when considering the morphism defined by the square of a line bundle on an irregular variety. Here we prove that

Proposition 3.8. *Let $a : X \to A$ be a generically finite morphism to an Abelian variety and V a Cartier divisor on X such that $\mathcal{O}_X(V)$ is full w.r.t. a (see Def. 2.3). Then $\mathcal{O}_X(V)$ is CGG at general points of X w.r.t. a, and the linear system $|2V|$ defines a generically finite map. In particular, the divisor V is big.*

Proof. If $a^* : Pic^0(A) \to Pic^0(X)$ is not an embedding, then we get a factorization

$$a = \iota \circ a' : X \to A' \to A,$$

where A' is the Abelian variety with dual $Pic^0(A') = a^*Pic^0(A) \subset Pic^0(X)$, and $\iota : A' \to A$ arises from the dual map $a^* : Pic^0(A) \to a^*Pic^0(A) = Pic^0(A')$. Note that $\iota^* : Pic^0(A) \to Pic^0(A')$ is finite and surjective, hence maps an open set to an open set. Since $H^0(X, F \otimes a^*\alpha) = H^0(X, F \otimes a'^*\iota^*\alpha)$ for a sheaf F on X and $\alpha \in \hat{A}$, we conclude that for a sheaf F on X,

$$F \ is \ CGG \ w.r.t. \ a \Leftrightarrow F \ is \ CGG \ w.r.t. \ a'$$

and

$$F \ is \ full \ w.r.t. \ a \Leftrightarrow F \ is \ full \ w.r.t. \ a'.$$

Then we consider the map a' instead, so we can assume $a^* : Pic^0(A) \to Pic^0(X)$ is injective in the following.

Consider the line bundle $\mathcal{L} := p_1^*\mathcal{O}_X(V) \otimes \mathcal{P}_a$ on $X \times \hat{A}$. Put

$$\hat{P} = Proj_{\mathcal{O}_A}(\oplus S^n((p_2)_*\mathcal{L})^*)$$

Then there exists an open set U of $\hat{P}$ parametrizes the divisors in $|\mathcal{O}_X(V) \otimes \alpha|, \alpha \in \hat{A}$. We can find an open set $U_0 \subset \hat{A}$ and a section $s : U_0 \to U$. Identifying U_0 with $s(U_0)$, we get a universal divisor

$$\mathcal{Y} \subset X \times U_0$$

and denote its closure in $X \times \hat{A}$ by $\mathcal{V}$. Then $\mathcal{V}_\alpha \equiv \mathcal{O}_X(V) \otimes \alpha$ for $\alpha \in U_0$,

As in [2] Sec. 5, take a general $\alpha_0 \in U_0$, and define a map

$$f_{\alpha_0} : Pic^0(A) \to Pic^0(X), \alpha \mapsto \mathcal{O}_X(\mathcal{V}_\alpha - \mathcal{V}_{\alpha_0})$$

which extends to a morphism. By rigidity,

$$f := f_{\alpha_0} - f_{\alpha_0}(0) : Pic^0(A) \to Pic^0(X)$$

is a homomorphism.

By definition, the image of f coincides with $a^*Pic^0(A)$. Identifying $\hat{A} = Pic^0(A)$ with $a^*Pic^0(A)$, we have $f = id_{\hat{A}}$. Then arguing as in Sec. 5.3 in [2], we prove

$$\mathcal{V} \equiv p_1^*\mathcal{O}_X(V) \otimes \mathcal{P}_a \otimes p_2^*\mathcal{O}_{\hat{A}}(\mathcal{V}_p)$$

where $p \in X$ is a point mapped to $0 \in A$ via a. Note that for $x \in X$, $\mathcal{V}_x = \{\alpha \in \hat{A} | x \in \mathcal{V}_\alpha\}$, and for two distinct points $x, y \in X$, if $a(x) \neq a(y)$, then $\mathcal{V}_x \neq \mathcal{V}_y$ since they are not even linearly equivalent. Since the map a is finite, considering the universal divisor $\mathcal{V}$ and arguing as in the proof of Lemma 3.3, we can show $|2V|$ defines a generically finite map.

Identifying $\hat{A}$ with a family of divisors on X via $\alpha \mapsto \mathcal{V}_\alpha$, then for a general point $x \in X$, $\mathcal{V}_x$ is a divisor on $\hat{A}$, which parametrizes the divisors passing through x. So it is easy to conclude that $\mathcal{O}_X(V)$ is CGG at general points of X. $\square$

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